Partial Restoration of Flavor Symmetry for Chiral Staggered Fermions [*]

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At non-zero lattice spacing the flavor symmetry of staggered fermions is broken to a discrete subgroup. We show that in the chiral limit the flavor symmetry of the pion effective Lagrangian enlarges to an $SO(4)$ subgroup of the continuum $SU(4)$ symmetry. This provides an explanation for observed degeneracies in the pion spectrum.

A single species of staggered fermion represents four degenerate quarks in the continuum limit, but at finite lattice spacing the $SU(4)$ flavor symmetry is broken down to a discrete subgroup[1]. We have found, however, that there is a substantial restoration of the flavor symmetry in the chiral limit, at least in the pion sector [2]. We give here a description of our conclusions and an outline of the arguments supporting them. Our analysis is an application of the method used in [3] to study flavor breaking with Wilson fermions.

We begin by recalling how flavor and rotation symmetries are broken by staggered fermions. In the continuum limit, the four flavors can be described by a field $Q_{\alpha,a}$, with spinor and flavor indices α and a, respectively. The Euclidean theory has an[3]

$$SU(4)_{\text{flavor}} \times SO(4)_{\text{rotation}} \qquad (1)$$

symmetry, under which the quarks transform as $Q \to (\Lambda_{1/2} \otimes U)Q$, with $\Lambda_{1/2}$ in the spinor rep. of $SO(4)$ and U an $SU(4)$ matrix. To obtain the subgroup of (1) respected on lattice, it is useful to consider first an intermediate subgroup, obtained by restricting flavor $SU(4)$ matrices to be those of the $SO(4)$ spinor rep:

$$U \longrightarrow \Lambda_{1/2} = \exp(i\omega_{\mu\nu}\sigma_{\mu\nu}). \qquad (2)$$

In addition, we keep a discrete set of matrices, $(i\gamma_\mu\gamma_5)$ which are contained in $SU(4)$, but not in

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[3]Other symmetries of the continuum theory will not be important here. Fermion number and the discrete symmetries (e.g. parity) are not broken by the discretization, while translation symmetry is broken in the obvious way.

$SO(4)$. These generate a 4-dimensional Clifford algebra, Γ_4, and rotate as a vector under $SO(4)$. Thus a subgroup of (1) is

$$\Gamma_4 \rtimes SO(4)_{\text{flavor}} \times SO(4)_{\text{rotation}} \qquad (3)$$

($\rtimes$ indicates a semidirect product).

Finally, consider the hypercubic subgroups of the two $SO(4)$ groups, which we refer to as SW_4 [4]. In particular, let $SW_{4,\text{diag}}$ be the hypercubic group in which flavor and spatial rotations are performed simultaneously. Then the flavor-rotation symmetry of staggered fermions is

$$\Gamma_4 \rtimes SW_{4,\text{diag}}. \qquad (4)$$

On the lattice the Γ_4 is generated by single-site translations, while the $SW_{4,\text{diag}}$ is composed of rotations. The details, which are not needed here, can be found in [1,4].

We can now state our result: in the chiral limit, at a fixed, but small, lattice spacing a, the effective flavor-rotation symmetry enlarges from the discrete group (4) to the continuous group (3). We have demonstrated that this restoration holds for the properties of pions, but suspect that it holds more generally.

Let us illustrate the consequences of this result for the pion spectrum. In the continuum limit, the pions lie in a 15-dimensional representation of $SU(4)$, created by the operators

$$\bar{Q}(\gamma_5 \otimes T^a)Q, \quad a = 1-15. \qquad (5)$$

A convenient choice of flavor matrices is

$$T^a = \{\xi_\mu, (i/2)[\xi_\mu, \xi_\nu], i\xi_\mu\xi_5, \xi_5\}, \quad \xi_\mu = \gamma_\mu^*, \quad (6)$$

and we use these (dropping factors of i and $i/2$) to label the states. On the lattice this 15-plet

breaks down into 7 irreps of the (discrete) symmetry group of the transfer matrix [a subgroup of the full lattice group (4)]. There are [5]

three $1-$d irreps : $\xi_5, \; \xi_4\xi_5, \; \xi_4,$ and
  four $3-$d irreps : $\xi_i\xi_5, \; [\xi_i,\xi_j], \; [\xi_4,\xi_j], \; \xi_i,$

where $i, j = 1-3$. At non-zero lattice spacing one expects, in general, that the masses of pions in the 7 irreps to be different. Our prediction is that, in the chiral limit, at non-zero but small a, the 7 irreps should collapse into the 4 irreps of flavor $SO(4)$, namely those with flavor

$$\xi_5, \; \xi_\mu\xi_5, \; [\xi_\mu,\xi_\nu], \; \xi_\mu . \tag{7}$$

There are thus three predicted degeneracies

$$M(\xi_4) = M(\xi_i), \qquad M(\xi_4\xi_5) = M(\xi_i\xi_5),$$
$$M([\xi_4,\xi_j]) \;=\; M([\xi_i,\xi_j]). \tag{8}$$

In more detail, the prediction is as follows. Assume that both $a^2\Lambda^2$ and m_q/Λ are small parameters (Λ is an abbreviation for $\Lambda_{\rm QCD}$), i.e. work close to the continuum and chiral limits. The general expansion of the pion masses is

$$\frac{M_\pi^2({\rm latt})}{\Lambda^2} \sim a^2\Lambda^2 + \frac{m_q}{\Lambda} + \frac{m_q}{\Lambda}a^2\Lambda^2 + \dots . \tag{9}$$

The m_q/Λ term is the leading continuum contribution, and respects the $SU(4)$ symmetry. The leading discretization errors are of two types: (1) The $a^2\Lambda^2$ term, which does not vanish in the chiral limit, and arises because the $SU(4)$ chiral symmetry is broken on the lattice even when $m_q = 0$; and (2) The $a^2 m_q \Lambda$ term, which vanishes in the chiral limit. Our result is that the former term respects the intermediate symmetry group (3), while the latter breaks the symmetry completely down to the lattice group (4). Thus in the chiral limit, for fixed a, the spectrum has an $SO(4)$ symmetry. Note that in this limit there is also an exact axial $U(1)$ symmetry, and so the $a^2\Lambda^2$ term is absent for the corresponding pion (which has flavor ξ_5).

How well does this prediction work? The complete spectrum of pions was first calculated with sufficient precision by JLQCD[6], and has recently been obtained for both unimproved and
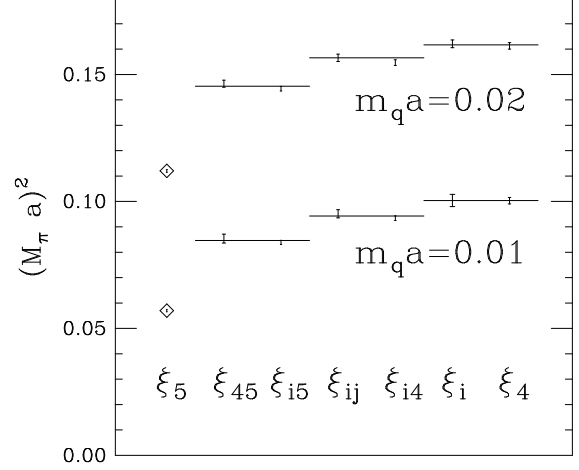

Figure 1. $M_\pi^2 a^2$ at $\beta = 6$ from Ref. [6]. Flavor is listed along the bottom, with the pseudo-Goldstone pion at the left. Horizontal lines show the average value of the mass-squared of the pairs of states which are predicted to become degenerate in the chiral limit.

improved staggered fermions by Orginos and Toussaint [7]. We plot the former results in Fig. 1. The quark masses correspond to roughly 1/3 and 2/3 of the strange quark mass. The predicted pattern is seen to hold: in most cases, the differences between the non-Goldstone pions in different $SO(4)$ irreps are significant, while pions in the same $SO(4)$ irrep have equal masses within errors. This pattern has apparently not previously been noted. The predictions are also well verified by the results of [7]. Apparently, one does not need to use extremely small quark masses in order to see the symmetry restoration.

We close this talk with an outline of our argument. As in [3] we proceed in two steps: (1) construct an effective continuum Lagrangian, $\mathcal{L}_{\rm eff}$ (in terms of quarks and gluons) describing the long-distance modes of the lattice theory; and (2) construct an effective chiral Lagrangian (in terms of mesons and baryons) describing the long-distance physics of $\mathcal{L}_{\rm eff}$. Following [8], $\mathcal{L}_{\rm eff}$ contains all operators allowed by the symmetries of the under-

lying lattice theory. It takes the form

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}} + a^2(\mathcal{L}_{\text{glue}}^{(6)} + \mathcal{L}_{\text{bilin}}^{(6)} + \mathcal{L}_{\text{ff}}^{(6)}) + O(a^4) \,, (10)$$

where the superscript indicates the dimension of the operators. The three dimension-6 terms contain gluonic, quark bilinear and four-fermionic operators, respectively. There are no fermionic operators of dimension 5 because of the symmetries of staggered fermions [9,10].

We are interested in the flavor and rotation symmetry breaking caused by the a^2 terms. The gluonic operators are flavor singlets, but break the rotation symmetry down to the hypercubic group: $SO(4) \rightarrow SW_4$. The fermion bilinears, which have been listed by Luo [11], turn out also not to break flavor. Thus flavor symmetry breaking is caused entirely by the four-fermion operators. Luo found 18 such operators; we have found 6 more.[4] These 24 operators divide into two types: (A) Lorentz singlets, which break flavor $SU(4) \rightarrow SO(4)$, e.g.

$$\mathcal{O}_1 = \sum_{\mu\nu} \left\{ [\bar{Q}(\gamma_\mu \otimes \xi_\nu)Q]^2 - [\bar{Q}(\gamma_\mu\gamma_5 \otimes \xi_\nu\xi_5)Q]^2 \right\}$$

(B) Operators which break the symmetry down to the discrete lattice subgroup (4), e.g.

$$\mathcal{O}_2 = \sum_{\mu} \left\{ [\bar{Q}(\gamma_\mu \otimes \xi_\mu)Q]^2 - [\bar{Q}(\gamma_\mu\gamma_5 \otimes \xi_\mu\xi_5)Q]^2 \right\}$$

The key point is that $\mathcal{O}_2$ is not invariant under separate spatial and flavor rotations. In fact, one can show that all type (B) operators can be chosen to be non-singlets under spatial rotations. In the present case $\mathcal{O}_2 - \mathcal{O}_1/4$ is such a non-singlet.

Now we proceed to the second step, and map $\mathcal{L}_{\text{eff}}$ into the chiral Lagrangian. The dimension-4 operators map as usual:

$$\mathcal{L}_{\text{QCD}} \rightarrow \Lambda^2 \text{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^\dagger) + m\Lambda^3 \text{Tr}(\Sigma + \Sigma^\dagger), (11)$$

where $\Sigma = \exp(i\pi^a T^a)$. The flavor singlet $O(a^2)$ terms give corrections of relative size $a^2 p^2$,

$$a^2 \mathcal{L}_{\text{glue+bilin}} \rightarrow a^2\Lambda^2 \sum_\mu \text{Tr}(\partial_\mu^2 \Sigma \partial_\mu^2 \Sigma^\dagger) + \ldots, (12)$$

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[4]Some of these 24 operators are redundant, but this overcounting takes care of itself automatically when we calculate physical quantities such as pion masses.

which break $SO(4) \rightarrow SW_4$. Four-fermion operators of type (A), being Lorentz singlets, can be mapped into operators in the chiral Lagrangian without derivatives, e.g.

$$\mathcal{O}_1 \rightarrow \sum_\nu \left\{ \text{Tr}(\Sigma\xi_\nu\Sigma^\dagger\xi_\nu) - \text{Tr}(\Sigma\xi_\nu\xi_5\Sigma^\dagger\xi_5\xi_\nu) \right\} . (13)$$

They thus contribute to the $a^2\Lambda^2$ term in (9), and break $SU(4) \rightarrow SO(4)$. The key point, however, is that, since type (B) operators are Lorentz nonsinglets, their chiral representatives must contain extra derivatives, e.g.

$$\mathcal{O}_2 - \mathcal{O}_1/4 \rightarrow \sum_\mu \left\{ \text{Tr}(\partial_\mu \Sigma\xi_\mu\partial_\mu\Sigma^\dagger\xi_\mu) \right.$$
$$\left. - \text{Tr}(\partial_\mu\Sigma\xi_\mu\xi_5\partial_\mu\Sigma^\dagger\xi_5\xi_\mu) \right\} . \quad (14)$$

Since $p^2 \rightarrow M_\pi^2 \sim m_q\Lambda$ on-shell, these operators only contribute to the $a^2 m_q\Lambda$ term in (9). They break the continuum symmetry down to the lattice group (4).

We conclude with some general comments. Our predicted symmetry restoration applies also for perturbatively improved staggered fermions, such as those studied in [7]. We stress, however, that we make no predictions for the values of the splittings between $SO(4)$ multiplets—these will depend on the discretization. Finally, as far as we can see, similar symmetry restoration should hold for other hadron masses.

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